

INVISIBLE PROOF: THE ROLE OF GESTURES AND ACTION IN PROOF

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The role of reasoning and proof in mathematics is undeniably crucial, and yet research in mathematics education has repeatedly indicated that students struggle with proof production. Our research shows that proof activities can be illuminated by considering action and gesture as a modality for crucial aspects of mathematical communication. We share two examples that highlight the importance of gesture and action in supporting students' mathematical proof production. We conclude by discussing the implications of our work for already existing schemes for classifying proof.

Keywords: Geometry and Geometrical and Spatial Thinking; Reasoning and Proof; Instructional Activities and Practices

Research in mathematics education has consistently shown that students, as well as pre-service and in-service teachers, struggle with constructing, interpreting, and evaluating proofs (Knuth, 2002; Healy & Hoyles, 2000; Chazan, 1993). Such research is deeply concerning, as proof is “an essential component of doing, communicating, and recording mathematics” (Schoenfeld, 1994, p. 74). Yet these results are perhaps unsurprising, as only recently has mathematics education begun to emphasize proof in the curriculum (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 2000). NCTM recommends that proof and reasoning be taught from prekindergarten to 12th grade. The reasoning and proof Process Standards most relevant to our research include: the investigation of mathematical conjectures; the development and evaluation of mathematical arguments and proofs; and the use of “various types of reasoning and methods of proof” (NCTM, 2000, p. 322). Building upon NCTM’s work, the new but widely-adopted *Common Core State Standards for Mathematics* identify constructing viable arguments and critiquing the reasoning of others as critical skills for students to learn across grade levels, while the standards for high school geometry specifically call for students to construct mathematical proofs of theorems (Common Core State Standards Initiative, 2010).

As the field continues to struggle with teaching proof, it is worth considering alternative forms of support for students. Healy and Hoyles (2000) found that correct proof production is easier for students when they can engage in building narrative forms, rather than algebraic ones. Various authors have examined the role of digital geometry environments in supporting the development of proof (Chazan, 1993; Hoyles & Jones, 1998). Here, we take a novel approach by looking to theories of embodied cognition, examining alternative methods of supporting proof production and communication through the modalities of action and gesture. We place ourselves on PME-NA’s learning continuum, as we are using an innovative method of examining *student learning* during proof production. This paper makes no claims as to the newness of action and gesture in supporting mathematical learning and communication—in fact, it is the ubiquitous yet overlooked role of action and gesture to which we wish to draw attention. Consequently, we contribute a new lens in order to reveal heretofore invisible proofs.

In this paper, we begin by briefly defining the practice of mathematical proof, and then explore relevant research on embodied cognition, action, simulated action, and gesture. We then share two excerpts of students using gesture and action to support proof from a recent study we conducted. In these excerpts

and others, we find that the practice of proof is greatly enriched by considering both verbal and physical modalities. Finally, we share our future plans for this research, and examine the potential implications for teaching proof.

Theoretical Framework

Mathematical Proof and Justification

We conceptualize mathematical justification using Harel and Sowder's (1998) proof scheme, and our intended "mode of thought" (e.g., the modality of mathematical observation and reflection; p. 240) is body-based action and gesture. Harel and Sowder define *proving* as "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). They further identify two subprocesses of proving: ascertaining (the proof activities an individual engages in when attempting to convince themselves); and persuading (the proof activities an individual engages in when attempting to convince others). As proof occurs in a social context where the argument must be communicated to an audience effectively and convincingly, we argue that each subprocess is essential when considering the learning of proof.

Harel and Sowder's (1998) proof scheme includes multiple categories and levels for classifying mathematical proofs. For our purposes, we focus on the *analytical>transformational* proof scheme, which involves "operations on objects and anticipations of the operations' results" (p. 259). In particular, when students are utilizing the analytical>transformational proof scheme, they are *transforming* a mathematical object or concept by varying some relationships purposefully in anticipation of certain results, observing the resulting changes, and deducing mathematical properties accordingly. Although this is a powerful and effective method of proving for students to learn, until now little research has examined how gesture and body-based action can play a role in supporting these dynamic transformations.

Gesture and Action

Theories of embodied cognition suggest that cognitive processes are not algorithms acting upon amodal mental systems, but rather they are bound up with the action and perception systems of the thinker (Barsalou, 1999; Barsalou, 2008; Glenberg & Robertson, 2000; Wilson, 2002). These action and perception systems, in turn, are not only guided by cognitive processes, but they also constitute and transform those processes. In other words, gestures and actions are not simply byproducts of cognition—they are coupled to cognitive processes (Shapiro, 2011) and they influence cognition. For example, gesture accompanied by speech may elaborate upon the thoughts possessed by the speaker (contributing additional information not contained by the speech acts), as well as feed back into processes that transform the speaker's cognition (Alibali & Kita, 2010; Goldin-Meadow & Beilock, 2010; Nathan & Johnson, 2012).

Gestures are a particular form of action that represent the world, rather than acting upon the world directly (Goldin-Meadow & Beilock, 2010). Furthermore, gestures are more than mere movements; as McNeill (1992) says, they "can never be fully explained in purely kinesthetic terms" (p. 105). Gesture is tied tightly to action, in our view, following Hostetter and Alibali's (2008) conceptualization of gestures as "manifestations of the simulated actions and perceptions that underlie thinking" (p. 508). Gestures are symbols that serve both to communicate and to affect the gesturer's cognition. Whether participants' gestures are produced as communicative or cognitive acts may appear to be a crucial distinction that we are in need of making. However, Hostetter and Alibali (2008) determine such a distinction to be somewhat false:

... gestures are a natural by-product of the cognitive processes that underlie speaking, and it is difficult to consider the two separately because both are expressions of the same simulation.... [G]esture and speech may express different aspects of that simulation ... but they derive from a single simulation; thus, they are part of the same system. (p. 508)

Consequently, we use verbal and gestural data side by side in our analyses for the purpose of triangulating on participants' cognition. By considering multiple modalities in this fashion, we are able to

gain access to elements of ascertaining (convincing oneself) and persuading (convincing others) proof activities that would otherwise remain hidden in plain sight.

Gesture, Action, and Mathematics

Learners' gestures and actions have been found to support mathematics learning in many previous studies (e.g., Glenberg et al., 2007; Nathan, Kintsch, & Young, 1992; Alibali & Nathan, 2012; Alibali & Goldin-Meadow, 1993), and are "involved not only in processing old ideas, but also in creating new ones" (Goldin-Meadow, Cook, & Mitchell, 2009, p. 271). In our research, we build upon this prior work while venturing into new territory: the role of action and gesture in supporting proof production. In the following section, we discuss gesture and mathematics in a general fashion, and draw out some threads that are particularly relevant to the domain of proof.

In some cases, gesture or body-based action may allow students to manipulate conceptual objects in a fashion similar to dynamic geometry software. In these systems, students can build objects that maintain invariant relationships even as the object is manipulated and acted upon (e.g., when a single vertex is moved on a triangle, the connected sides will stretch to meet the new location of the vertex, always keeping a triangular shape) (Hoyles & Jones, 1998). These environments can support students in generating and verifying conjectures about the relationships contained within these objects. Similarly, the real-world context within which action and gesture are produced can give feedback about the legitimacy of the constructs evoked by the mathematical conjectures. Whereas with paper and pen impossible triangles can be constructed (e.g., a triangle where the hypotenuse is labeled as longer than the sum of the remaining two sides; see Table 1 for the relevant task), using one's body to construct the triangle can constrain a student from expressing such an impossibility.

One of the features of dynamic geometry software is that it affords the possibility of testing a large number of examples while maintaining the relevant invariant features. Gesture and simulated action may have a similar affordance. Hostetter and Alibali (2008) note that, "Because mental images retain the spatial, physical, and kinesthetic properties of the events they represent, they are dependent on the same relationships between perceptual and motor processes that are involved in interacting with physical objects in the environment" (p. 499). In other words, mental simulation, as evidenced through action and gesture, also allows such testing. We hypothesize that action and gesture can influence cognition in a way that is soundly based upon the physical experience of space within the world. In this way, action and gesture may support analytical>transformational proof production.

Given the potential influence of action, gesture, and simulated action upon cognition, we designed an experiment to examine the role of action and gesture upon mathematical proof production and communication. In the following section, we share our methodology and mathematical conjecture tasks.

Methods

The data reported in this paper were drawn from a larger study of the role of action and gesture in proof. Participants were 36 students (22 F; 14 M) at a large Midwestern university enrolled in a psychology course, and they received partial class credit for participating. The average age of our participants was 20 years old (15 freshmen; 9 sophomores; 7 juniors; 4 seniors; 1 part-time student).

In one-on-one interviews, participants were asked to justify a variety of mathematical theorems from number theory and geometry; here we report on the two conjectures shown in Table 1. Our research question was: *How are gestures and actions used in the ascertaining and persuading phases of proof?* In the larger study (not reported here), we encouraged participants to produce particular sorts of gestures for some conjectures. However the analysis here focuses only on cases in which students spontaneously gestured and used action when proving (i.e., their behavior was not manipulated).

Table 1: Conjectures

Conjecture 1: Gears	Conjecture 2: Triangle Inequality
<i>An unknown number of gears are connected in a chain. You know which direction the first gear turns. How can you determine what direction the last gear will turn? Provide a justification for your answer.</i>	<i>Mary came up with the following conjecture: "For any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side." Provide a conceptual justification as to why Mary's conjecture is true or false.</i>

We intentionally created a context in which participants would be likely to produce gestures while reasoning mathematically by setting up the discourse context (Hostetter & Alibali, 2008) in various ways. First, we emphasized the talk-aloud nature of the experiment to all of our participants, and we verbally prompted them to reason out loud if they fell silent. For some, we removed pen and paper; and for others we also removed the chair so that they (and the interviewer) had to remain standing. Furthermore, by designing the protocol to give no feedback to the participants, we emphasized *innovative* proof instead of *imitative*, and shifted the mode of thought to mental and physical by lessening the availability of pen and paper.

Each participant was videotaped, with the camera(s) capturing both small and large gestures. The data were analyzed in Transana, a qualitative analysis software program, and team members selected various gesture segments with the units of analysis based on each mathematical conjecture. In particular, we used *multimodal analysis* techniques (Alibali & Nathan, 2012; McNeill, 1992), and coded how each student used action and gesture during the ascertaining and persuading phases of their proof (Harel & Sowder, 1998).

Results

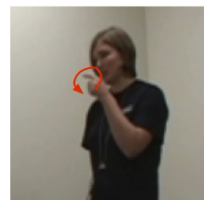
Our first example shows how simulated action (manifested through gesture) can illuminate the ascertaining phase of proof. The second example demonstrates the role of paired gesture and speech during the persuading phase. Our presentation we will provide additional examples.

Simulated Action Illuminates Ascertaining

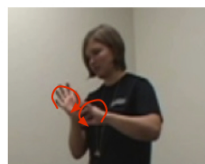
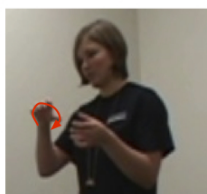
To illustrate the illumination of proof by simulated action, we share an excerpt from the Gears conjecture (Table 1), as the participant leverages her body as a tool for simulating the actions of the gears and identifying parity (shown in Figure 1). The excerpt contains both ascertaining and persuading phases of the proof, and is annotated as such.

- 1 Well, if you know which direction the first gear is turning, then you also know that the second gear must be turning in the opposite direction,**

(Ascertaining) Uses her hands to replicate the turning of a gear

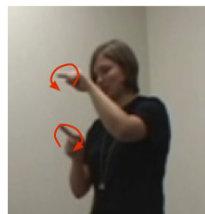
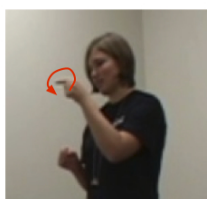


- 2 Because one gear turning would push the other gear...**



(Ascertaining) Uses both her hands to simulate the movement of two gears turning against each other

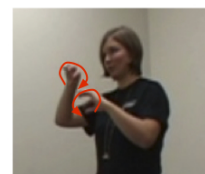
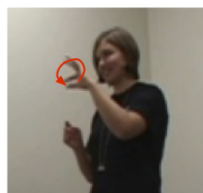
- 3 See if one gear was turning this way, then the spokes on it would push...**



(Ascertaining) Uses her hands to simulate the spokes of one gear pushing the second gear

- 4 If one gear was turning like that, then the spokes would push the other gear in the opposite direction, so you would know that every other gear from the first gear was turning the same direction as the first gear.**

(Persuading) Uses her hands to simulate the movement of two gears as she solves the conjecture.



- 5 So all the odd gears would be turning in the original direction and all the even gears would be turning in the opposite direction. So if the last gear was odd it would be turning in the original direction and if the last gear was even it would be turning in the opposite of the original direction.**

Figure 4: Simulated action illuminates ascertaining

Paired Gesture and Speech Persuading

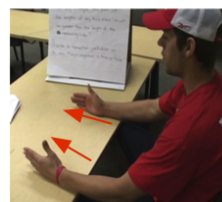
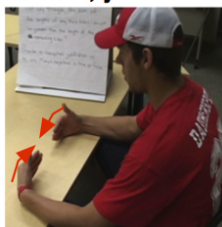
To illustrate the pairing of gesture and speech to support persuading in embodied proof, we examine an excerpt that occurs after a participant has solved the Triangle Inequality conjecture (Table 1) and has shifted into the persuading phase of the proof (Figure 2). The verbal element of the proof provides a specific example, as simultaneously the gestural components communicate the generalizability of the participant's proof.

1 Say one side's 5.

Uses his fingers to indicate a general side length

2 If the other two sides weren't at least 5 or even equal to 5,

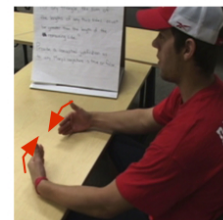
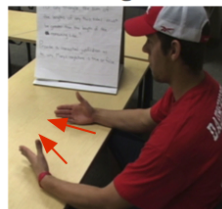
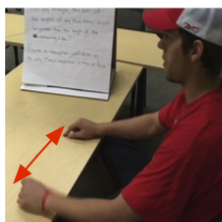
Uses his hands to indicate the other two sides, of unspecified lengths but shorter than the general side length indicated in Line 1

**3 if you tried to connect them, you think about it, they'd be too short to touch.**

Curves his hands in to illustrate the two shorter sides trying to close the triangle

4 So they would have to-- the two sides would have to be longer

Repeats the gesture of the two shorter sides failing to meet; re-invokes the original conjecture

**5 than the other remaining side.**

Repeats the gesture of the first general side length; completes answering the conjecture.

Figure 2: Paired gesture and speech persuading

In Figure 2, the verbal and gestural components are woven together to provide a complete proof. Attending only to the verbal proof elements would result in an incomplete empirical justification, as the participant would appear to be basing his entire argument upon testing a single (and incompletely described) triangle. However, in considering the gestures, we gain insight into the participant's full argument that goes beyond empirical, into analytical and even axiomatic proof schemes—the realms of a mathematically legitimate proof. It is through multimodal communication of gesture paired with speech that the student presents the most compelling and persuasive case for supporting his conjecture.

Discussion

These two examples highlight the multi-modal nature of proof, and show that understanding proof production can require attending to more than just students' verbal and written work. In Figure 1, the participant's embodied account reveals how she relies on an early instance to establish a conjecture about gears that is sufficiently general to support a deductive proof scheme. In Figure 2, the persuading phase

offered might seem superficial (overly empirical) as a strictly verbal account. However, the participant's accompanying gestures reveal a corroborative proof scheme that is analytical in the sense that it relies, not on the particular lengths or topology, but their structural role. At the same time, it is transformational in how it utilizes actions to support the goal of portraying the impossibility of any triangle that rejects the premises.

These are not rare examples from our data, but rather they are characteristic of many other proof schemes we observed. Alongside our exploration of the different modalities of proof, we are examining various ways that gesture and action can support mathematical learning and, consequently, proof production. Although the data reported here come from proofs that participants spontaneously generated, interventions that manipulate action and gesture show promise for supporting analytical>transformational proof production (Walkington et al., 2012). As many students have difficulty producing traditional deductive proofs, preferring inductive empirical reasoning (Chazan, 1993; Healy & Hoyles, 2000; Hoyles & Healy, 1999, 2007), gestures and actions may provide an accessible bridge between the two. The potential of simple physical movements to support mathematical understanding is vast—and a crucial new area of study, given the importance of proof to the mathematical community and the general difficulty of engaging students in proof practices.

Conclusions

The implications of this emerging research on embodied cognition are profound for mathematics education in general, and the teaching of mathematical proof in particular. Action and gesture provide another modality for mathematics learning and expression, which may particularly support those students who struggle with the abstract notation traditionally used with proof. Extending the examination of proof production into gesture and action allows us to conceptualize a more complete model of cognition (Shapiro, 2011), and consequently allows us to design new activities that more coherently account for different strategies of proof production.

Our research provides a starting point for those examining mathematical proof through the modalities of action and gesture, and we continue to research the impact of action and gesture upon proof production. This work raises an important question: How does an embodied account influence earlier proof frameworks (e.g., Harel & Sowder, 1998; Healy & Hoyles, 2000)? Our next step is to answer exactly that question, and provide a multidimensional framework that incorporates a proof scheme with a spectrum of gesture, action, and proof.

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